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Note

An upper bound on the sum of squares of degrees in a graph

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Abstract

Let G be a simple graph with n vertices, e edges and vertex degrees $d_1, d_2, \ldots, d_n$. It is proved that $d_1^2 + \cdots + d_n^2 \leqslant e(2e/(n-1) + n - 2)$ when $n \geqslant 2$. This bound does not generalize to all sequences of positive integers. A comparison is made to another upper bound on $d_1^2 + \cdots + d_n^2$, due to Székely et al. (1992). Our inequality follows from the positive semidefiniteness of a certain quadratic form in $\binom{n}{2}$ variables. We also apply the inequality to bounding the total number of triangles in a graph and its complement. © 1998 Elsevier Science B.V. All rights reserved

1. Introduction

Throughout this paper $G=(V,E)$ will denote a simple graph with n vertices and e edges. To avoid trivialities we always assume that $n \geqslant 2$. Also, d_i is the degree of the ith vertex.

Theorem 1.

$$d_1^2 + \cdots + d_n^2 \leqslant e\left(\frac{2e}{n-1} + n - 2\right). \tag{1}$$

This will be proved in the next section. Here we make comparisons to other known results. The Cauchy–Schwarz inequality yields the lower bound

$$d_1^2 + \cdots + d_n^2 \geqslant \frac{1}{n}(d_1 + \cdots + d_n)^2 = \frac{4e^2}{n}. \tag{2}$$

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Recall that $\tilde{d} := 2e/n$ is the average degree. There is also the well-known identity [3, Exercise 10.30]

$$d_1^2 + \cdots + d_n^2 = \sum_{ij \in E} (d_i + d_j). \tag{3}$$

(For brevity we write ij for the two-element set $\{i,j\}$.) Thus, (2) may be interpreted as saying that the average value of $d_i + d_j$, as ij ranges over edges, is at least $2\tilde{d}$. On the other hand, (1) implies that this same average is at most $\tilde{d} + n - 1$. (It is easy to check that $2e/(n-1) + n - 2 \leqslant 2e/n + n - 1$ for any simple graph.) This may be intuitively plausible; and the same intuition might lead one to conjecture that

$$\frac{1}{e} \sum_{ij \in E} (d_i + d_j) \underset{?}{\leqslant} \tilde{d} + \Delta, \tag{4}$$

where Δ is the maximum degree. However, (4) is false, in general, as the following example shows. Let G consist of K_r plus r isolated vertices. For this example the average value of $d_i + d_j$, $ij \in E$, is $2(r-1)$, whereas $\tilde{d} + \Delta$ equals $\frac{3}{2}(r-1)$, so that (4) is not valid.

There is a quite different upper bound on the sum of squares of degrees, due to Székely et al. [5]

$$d_1^2 + \cdots + d_n^2 \leqslant (\sqrt{d_1} + \cdots + \sqrt{d_n})^2. \tag{5}$$

The inequalities (1) and (5) are incomparable: in the example above, (5) performs better than (1). However, when $G = K_{s,s}$ then $d_1^2 + \cdots + d_n^2$ equals $2s^3$; (1) yields an upper bound of approximately $3s^3$; and the right-hand side of (5) equals $4s^3$. In general, the bound (1) is perhaps a bit more useful than (5), since it depends only on n and e rather than the full-degree sequence. By Cauchy–Schwarz, $(\sqrt{d_1} + \cdots + \sqrt{d_n})^2 \leqslant 2en$; but the inequality $d_1^2 + \cdots + d_n^2 \leqslant 2en$ is weaker than (1), and is, in fact, a trivial upper bound. Indeed, by (3) it asserts that the average of $d_i + d_j$, as ij varies over edges, is at most $2n$; since vertex degrees in a simple graph are at most $n-1$, this is obvious.

To conclude this introduction, we note that (1) is not valid for all sequences $d_1, \ldots, d_n$ of positive integers (adding up to an even integer $2e$). The reader can easily supply suitable examples. This suggests that an elementary inductive proof of (1) may be difficult.

2. A quadratic form

Let $[n] = \{1, 2, \ldots, n\}$ be the canonical n-element set, and let $[n]^{(2)}$ denote the set of two-element subsets of $[n]$ (or, if one prefers, the edge-set of K_n). To each $ij = \{i,j\}$ in $[n]^{(2)}$ associate a real variable x_{ij}; these variables are assumed to be algebraically independent.

Theorem 2. *For $n \geqslant 2$, and for all real x_{ij}'s*

$$\left(\sum_{ij} x_{ij}\right)^2 + \binom{n-1}{2}\sum_{ij} x_{ij}^2 - \frac{(n-1)}{2}\sum_i \left(\sum_{j \neq i} x_{ij}\right)^2 \geqslant 0. \tag{6}$$

Proof. We wish to show that the quadratic form on the left-hand side of (6) is positive semidefinite. Expanding squares and collecting terms, one easily finds that this quadratic form equals

$$Q = \binom{n-2}{2}\sum_{ij} x_{ij}^2 - (n-3)\sum_{\{ij,ik\}} x_{ij}x_{ik} + 2\sum_{\{ij,kl\}} x_{ij}x_{kl}, \tag{7}$$

where the second sum is over all unordered pairs of incident distinct 2-element subsets of $[n]$, and the third sum is over all unordered pairs of disjoint 2-subsets. The associated symmetric matrix is then

$$\begin{aligned} S &= \binom{n-2}{2} I - \frac{1}{2}(n-3)A + (J - I - A) \\ &= \frac{1}{2}(n-1)(n-4)I + J - \frac{1}{2}(n-1)A, \end{aligned} \tag{8}$$

where $A = A(L(K_n))$ is the adjacency matrix of the line-graph of K_n, and I and J are the identity and all-ones matrix of order $\binom{n}{2}$. The eigenvalues of A are well known and easy to calculate (cf. [3, Example 11.2]): $2(n-2)$ (which is the line-sum of A), $(n-4)$ and -2, with suitable multiplicities. Since I, J and A are pairwise commuting symmetric matrices, they have common eigenspaces and hence the eigenvalues of S are easily calculated by adding matching eigenvalues of the three summands in (8). The result is that S has just the two distinct nonnegative eigenvalues 0 and $\frac{1}{2}(n-1)(n-2)$; so S is positive semidefinite. □

Theorem 1 follows easily from Theorem 2. Indeed, given a simple graph G on $n \geqslant 2$ vertices and with e edges, let $x : [n]^{(2)} \to \mathbb{Q}$ be the indicator function for E, that is $x_{ij} = 1$ if ij is an edge and $x_{ij} = 0$ otherwise. Clearly, $\sum_{ij} x_{ij} = \sum_{ij} x_{ij}^2 = e$, and for each i, $\sum_j x_{ij} = d_i$. Hence, (1) follows from (6), applied to this particular vector x.

We conclude with a remark on possible applications of (1). Let $t(G)$ denote the number of triangles in G. It was first observed by Goodman [1] that $t(G) + t(\overline{G})$, where $\overline{G}$ denotes the complement of G, is determined by the degree sequence:

$$t(G) + t(\overline{G}) = \frac{1}{2}\sum_{i \in V}\left[d_i - \frac{(n-1)}{2}\right]^2 + \frac{n(n-1)(n-5)}{24}. \tag{9}$$

From (9) we get $t(G) + t(\overline{G}) \geqslant \frac{1}{24}n(n-1)(n-5)$, with equality iff n is odd and G is regular of degree $\frac{1}{2}(n-1)$. Goodman [2] raised the question of finding a best possible upper bound of the form $t(G) + t(\overline{G}) \leqslant B(n,e)$, and he explicitly conjectured an expression for $B(n,e)$. This conjecture was proved recently by Olpp [4]. We remark that a nontrivial upper bound on $t(G)+t(\overline{G})$ follows from (9) and (1), although it falls short of the true optimum. For example, if e is approximately $\frac{1}{2}\binom{n}{2}$, then (9) and (1) together imply that $t(G)+t(\overline{G})$ is at most roughly $\frac{5}{8}\binom{n}{3}=0.625\binom{n}{3}$; we omit the straightforward calculation. By contrast, an essentially optimal example is obtained by taking a complete graph with about $n/\sqrt{2}$ vertices (so $e\simeq\frac{1}{2}\binom{n}{2}$); in this case $t(G)+t(\overline{G})$ is roughly $0.561\binom{n}{3}$ Note that Olpp's theorem, in conjunction with (9), implies a best possible upper bound on $d_1^2+\cdots+d_n^2$, in terms of n and e only; however, the expression of this bound is rather complicated. The weaker estimate (1) is still nontrivial and has an appealingly simple form.

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